# Readers' Forum

*Brief discussions of previous investigations in the aerospace sciences and technical comments on papers published in the* AIAA Journal *are presented in this special department. Entries must be restricted to a maximum of 1000 words, or the equivalent of one* Journal *page including formulas and figures. A discussion will be published as quickly as possible after receipt of the manuscript. Neither the AIAA nor its editors are responsible for the opinions expressed by the correspondents. Authors will be invited to reply promptly.*

## Comment on "Unconstrained Variational Statements for Initial and Boundary Value Problems"

Cecil D. Bailey*
*The Ohio State University, Columbus, Ohio*

I APPRECIATE Dr. Simkins' reference[1] to two of my recent papers.[2,3] However, his claim that I use an "unconstrained variational statement" is erroneous. I assume that Dr. Simkins is not aware of the discussion of Ref. 3[4] and the comments[5-7] on two other of my papers.[8,9]

The authors of Ref. 5 state, "This is wrong" and, "There are internal inconsistencies..." The author of Ref. 6 states, "We believe the theory is basically incorrect," and correctly points out that I do not "define the work function $W$ so that

$$\delta \int_{t_0}^{t_1} W \mathrm{d}t$$

is a meaningless expression."[6] The author of Refs. 4 and 7 write, "There are several errors in concept which need to be corrected to place the results on a good theoretical foundation." He then proceeds to state that the equations,

$$\delta \int_{t_0}^{t_1} (T+W)\mathrm{d}t - \Sigma\left(\frac{\partial T}{\partial \dot{q}_i}\right)\delta q_i \Big|_{t_0}^{t_1} = 0$$

which Hamilton called the "Law of Varying Action,"[10] "is not a correct statement, by any name, for nonconservative systems since the $W$ function simply does not exist."[4]

My replies to these comments are now published.[11-14] Therein, I point out that it is precisely the formulation, in accordance with my concepts of spacetime continuity, which led me to the foregoing equation and the "really outstanding numerical results"[4] which are presented in our papers.[2,3,8,9,15-20] I won't belabor the points already made in my replies to these comments; but, contrary to the claim of "does not exist" made by the author of Refs. 4 and 7, the direct analytical solution to a nonconservative follower force system directly from this "nonexistent" equation is presented in Ref. 14.

I am puzzled as to why the existence of the equation of which Hamilton wrote, "A different estimate, perhaps, will be formed of that other principle which has been introduced in the present paper, under the name of the law of varying action... the peculiar combination which it involves, of the principles of variations with those of partial differentials, for the determination and use of an important class of integrals, may constitute, when it shall be matured by the future labors of mathematicians, a separate branch of analysis,"[10] is not recognized by the author of Refs. 4 and 7 while Dr. Simkins concludes that it is an "unconstrained variational statement,"[1] which, of course, implies that it does "exist." It is interesting and, in my opinion, significant, that no such diametrically opposed conclusions have been advanced by anyone of the many eminent people with whom I have corresponded on this subject.

Dr. Simkins writes, "...unconstrained variational formulations for initial value problems... was first shown by Tiersten."[1] Professor Tiersten's extension of Hamilton's *principle* and Dr. Simkins application of the Lagrange multiplier with equations of constraint accomplish with complexity that which can be accomplished with simplicity by use of my concepts applied to Hamilton's *law*. Nowhere in our papers will one find a Lagrange multiplier, a Dirac delta function, reference to singular points, or the requirement of a potential function as a necessary condition. One simply defines the system and generates the solution.

Dr. Simkins also writes, "In spite of the apparent generality of the Lagrange multiplier method, its application to Hamilton's principle (a constrained variational principle) for the solution of initial value problems is not obvious."[1] In 1962, Bisplinghoff and Ashley put it a different way: "The question of how to handle the upper limit $t_1$ during direct application of Hamilton's principle is a more subtle one."[21] According to Dr. Simkins, "Tiersten's procedure requires a special introduction of one of the initial conditions into the variational statement, making incomplete use of the Lagrange multiplier method in the time domain."[1] He further notes that the success of Tiersten's method for "achieving solutions to initial value problems was never tested."[1] With the complexity as now employed by Dr. Simkins, it is not difficult to imagine why the method was "never tested."

Dr. Simkins makes reference to my papers; "indeed, Bailey found it fruitful to employ Hamilton's law of varying action in which no functional exists." This implies that Dr. Simkins feels that there is no question except that my theory is based in the calculus of variations. Such is not the case. As I have pointed out elsewhere,[15] my use of the symbol $\delta$ does not mean that I am using the concepts of the calculus of variations. I first called the equation the "general energy equation"[22] because, it obviously, was not Hamilton's *principle*. On reading Hamilton's original papers[10,23] in 1973, I was astounded by the statements which he made. My search of the literature at that time did not reveal anyone who had ever employed Hamilton's *law*. No one to date has come forward with evidence of such application. My search led directly to Lagrange and the concepts which underlie the rigorous proof of the principle of least action as the source of the confusion[6] which surrounds Hamilton's principle and which is the reason why we had no exposure to Hamilton's papers except through the concepts associated with Hamilton's principle.

In example 1 of his paper, Dr. Simkins presents a solution to the wave equation. He deduces a variational statement, his Eq. (16), with Lagrangian multipliers and equations of constraint. Such complexity is unnecessary. The differential equation for the wave as presented by Dr. Simkins is identical to the differential equation for the torsional motion of an elastic rod. The direct analytical solution may be obtained without any reference to a differential equation or to the use of Lagrange multipliers.

Received Aug. 10, 1978; revision received Oct. 30, 1978. 
Index category: Structural Dynamics.
*Professor. Member AIAA.

Dr. Simkin's example 2, the linear oscillator, was demonstrated in Ref. 2 without the necessity of Lagrange multipliers and equations of constraint. The direct analytical solution to a nonlinear oscillator is presented in Ref. 15. Example 3, of course, may be formulated and the direct analytical solution obtained through Hamilton's law without use of Dirac functions and without use of Lagrange multipliers.

It is significant that all three of Dr. Simkins' examples treat conservative systems. Direct analytical solutions to both conservative and nonconservative systems, both discrete[2,8,15] and for continuua[3,9,14,24] have now been demonstrated through application of Hamilton's law. With this law, it makes little difference whether the system is conservative or nonconservative (both conservativeness and stationarity were *assumed* by Lagrange in order to produce his rigorous proof of the principle of least action).

I must again express my appreciation to Dr. Simkins for his reference to my papers. However, in my opinion, his generation of "unconstrained variational statements" fails on at least two counts, simplicity and generality.

## References


[1] Simkins, T. E., "Unconstrained Variational Statements for Initial and Boundary-Value Problems," *AIAA Journal*, Vol. 16, June 1978, pp. 559-563.

[2] Bailey, C. D., "The Method of Ritz Applied to the Equation of Hamilton," *Computer Methods in Applied Mechanics and Engineering*, Vol. 7, Feb. 1976, pp. 235-247.

[3] Bailey, C. D., "Hamilton, Ritz and Elastodynamics," *ASME Journal of Applied Mechanics*, Vol. 43, Dec. 1976, pp. 684-688.

[4] Smith, C. V. Jr., "Discussion of 'Hamilton, Ritz and Elastodynamics'," *ASME Journal of Applied Mechanics*, Vol. 44, Dec. 1977, pp. 796-797.

[5] Smith, C. V. Jr. and Smith, D. R., "Comment on 'Application of Hamilton's Law of Varying Action'," *AIAA Journal*, Vol. 15, Feb. 1977, pp. 284-286.

[6] Bryson, A. E. Jr., "Comment on 'Application of Hamilton's Law of Varying Action'," *AIAA Journal*, Vol. 14, Sept. 1976, p. 1341.

[7] Smith, C. V. Jr., "Comment on 'Exact and Direct Analytical Solutions to Vibrating Systems with Discontinuities'," *Journal of Sound and Vibration*, Vol. 54, Oct. 1977, pp. 454-459.

[8] Bailey, C. D., "Application of Hamilton's Law of Varying Action," *AIAA Journal*, Vol. 13, Sept. 1975, pp. 1154-1157.

[9] Bailey, C. D., "Exact and Direct Analytical Solutions to Vibrating Systems with Discontinuities," *Journal of Sound and Vibration*, Vol. 44, Jan. 1976, pp. 15-25.

[10] Hamilton, W. R., "On a General Method in Dynamics," *Philosophical Transactions of the Royal Society*, Vol. 124, 1834, pp. 247-308.

[11] Bailey, C. D., "Reply by Author to C. V. Smith Jr., and D. R. Smith," *AIAA Journal*, Vol. 15, Feb. 1977, pp. 286-287.

[12] Bailey, C. D., "Reply by Author to A. E. Bryson, Jr.," *AIAA Journal*, Vol. 14, Sept. 1976, p. 1341.

[13] Bailey, C. D., "Reply by Author to C. V. Smith Jr.," *Journal of Sound and Vibration*, Vol. 54, Oct. 1977, pp. 454-459.

[14] Bailey, C. D., "Closure to Discussion by C. V. Smith, Jr.," *ASME Journal of Applied Mechanics*, Vol. 44, No. 4, Dec. 1977, pp. 796-797.

[15] Bailey, C. D., "A New Look at Hamilton's Principle," *Foundations of Physics*, Vol. 5, Sept. 1975, pp. 433-451.

[16] Bailey, C. D., "Direct Analytical Solutions to Nonuniform Beam Problems," *Journal of Sound and Vibration*, Vol. 56, April 1978, pp. 501-507.

[17] Beres, D. P., "Vibration Analysis of a Completely Free Elliptical Plate," *Journal of Sound and Vibration*, Vol. 34, June 1974, pp. 441-443.

[18] Beres, D. P., and Bailey, C. D., "Vibration of Skewed Cantilever Plates and Helicoidal Shells," NASA CR-2588, Sept. 1975.

[19] Ogg, J. S., "Vibrational Characteristics of Cracked Cantilever Plates," USAF-ASD-TR-77-65, Oct. 1977.

[20] Bailey, C. D., "Vibration and Local Edge Buckling of Thermally Stressed, Wedge Airfoil Cantilever Wings," Paper No. 73-327, AIAA Dynamics Specialist Conference, Williamsburg, Va., March 1973.

[21] Bisplinghoff, R. L. and Ashley, H., *Principles of Aeroelasticity*, John Wiley and Sons, Inc., New York, 1962, p. 36.

[22] Bailey, C. D., "Application of the General Energy Equation: A Unified Approach to Mechanics," Final Report, NASA Grant NGR 36-008-197, Aug. 1975.

[23] Hamilton, W. R., "Second Essay on a General Method in Dynamics," *Philosophical Transactions of the Royal Society*, Vol. 125, 1835, pp. 95-144.

[24] Bailey, C. D. and Haines, J. L., "The Hamilton-Ritz Formulation Applied to the Vibration and Stability of Non-Conservative Follower Force Systems," submitted to *Computer Methods in Applied Mechanics and Engineering*, Sept. 1978.


# Reply by Author to C. D. Bailey


T. E. Simkins*

*U.S. Army Armament Research and Development Command, Benet Weapons Laboratory, LCWSL, Watervliet, N.Y.*


A BRIEF review may help to clarify the relationship of my own work[1] to that of Professor Bailey.[2] Hamilton's principle may be derived in a few steps from D'Alembert's principle. The next to last step in this derivation yields the variational statement[3]:

$$\int_{t_1}^{t_2} (\delta T+\delta \bar{W})\,\mathrm{d}t-\left[\sum_{i=1}^{N} m_i\dot{q}_i\delta q_i\right]_{t_1}^{t_2}=0 \tag{1}$$

$T$ is the kinetic energy and $\delta\bar{W}$ is the virtual work done by the impressed forces on a system of $N$ particles having mass $m_i$ and coordinates $q_i$. The bar indicates that, in general, all of the terms of $\delta\bar{W}$ are not derivable from a scalar function. Now Eq. (1) is a valid physical statement that can be employed, as Bailey has done, as a basis for achieving approximate solutions to dynamics problems. For these purposes one need not proceed any further into the derivation.

Hamilton, of course, went on to require that $\delta q_i(t_1)=\delta q_i(t_2)=0$ and $\delta\bar{W}=\delta W=-\delta V$; i.e., the virtual work term is such that it can be expressed as the variation of a single scalar function. The result is Hamilton's *principle*:

$$\delta\int_{t_1}^{t_2} (T-V)\,\mathrm{d}t=0 \tag{2}$$

One notes in passing that Lanczos[4] has shown that even when $V$ is explicitly time variant, Eq. (2) still applies. Thus Hamilton's principle is valid for a limited class of nonconservative systems.

In the event the second requirement is removed, the result is called Hamilton's *extended principle*[3]:

$$\int_{t_1}^{t_2} (\delta T+\delta\bar{W})\,\mathrm{d}t=0 \tag{3}$$

Note that neither Eq. (1) nor Eq. (3) is of the form $\delta I=0$; hence neither is a "principle" in the same sense that Eq. (2) is—they might better be called *variational statements*. This does not, however, prevent either from being used as a basis for achieving approximate solutions to many problems in mechanics. In fact, Eq. (3) is the most widely used form of Hamilton's principle for such applications—Smith's comments notwithstanding.[5]


Received Dec. 6, 1978. 

Index category: Structural Dynamics.

*Research Mechanical Engineer.